Komplexní analýza - cvičení

Alexandra Olivová

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Úkol 1: $\forall z \in \mathbf{C}\ \exists y \in \mathbf{C} : y^2 = z.$

Řešení.
$y^2 = (a+bi)^2 = (a+bi)(a+bi) = a^2+2abi+b^2i^2 = a^2+2abi-b^2 = (a^2-b^2)+i2ab = z.$

Úkol 2: *Vyjádřete* $z \in \mathbf{C}$ *pomocí lineární kombinace funkce Im.*

Řešení.
$z = a + ib$
$a = Im(iz) = Im(ia + i^2b) = Im(ia - b),$
$ib = Im(z),$
$z = Im(iz) + Im(z).$

Úkol 3: *Dokažte, že platí* $[f(g(z))]' = f'(g(z))g'(z).$

Řešení.
Definice derivace: $f'(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$
$[f(g(z))]' = \lim_{z\to z_0} \frac{f(g(z))-f(g(z_0))}{z-z_0} \cdot \frac{g(z)-g(z_0)}{g(z)-g(z_0)} = \lim_{z\to z_0} \frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0)} \cdot \frac{g(z)-g(z_0)}{z-z_0} =$
$\lim_{z\to z_0} \frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0)} \cdot \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = f'(g(z))g'(z).$

Úkol 4: *Spočtěte.*

- $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i.$
- $(3+4i)^3 = (3+4i)(3+4i)(3+4i) = (9+24i-16)(3+4i) = (24i-7)(3+4i) = (72i - 96 - 21 - 28i) = 44i - 117.$
- $\left(\frac{2+2i}{2-2i}\right)^{19} = \left(\frac{2+2i}{2-2i} \cdot \frac{2+2i}{2+2i}\right)^{19} = \left(\frac{(2+2i)^2}{4+4}\right)^{19} = \frac{(2+2i)^{38}}{8^{19}} = \frac{(2+2i)^{38}}{2^{57}} = \frac{(2+2i)^{38}}{2^{57}} = \frac{-i2^{57}}{2^{57}} = -i.$

  Pomocné výpočty:
  $r = |z| = |2 + 2i| = \sqrt{8} = 2\sqrt{2},$
  $\arctan \frac{b}{a} = \arctan 1 = \frac{\pi}{4},$
  $z^{38} = (2 + 2i)^{38} = (2\sqrt{2})^{38} \cdot [\cos 38 \cdot \frac{\pi}{4} + i \sin 38 \cdot \frac{\pi}{4}] =$
  $= 2^{38} \cdot 2^{19} \cdot [\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}] = -2^{57}i.$

Úkol 5: *Zakreslete v komplexní rovině následující množiny.*

- $|z - 5i| \leq 5$,
- $|z + i| = |z - i|$.

Poznámka: obrázky.

Úkol 6: *Rozhodněte, zda jsou následující funkce holomorfní a případně v jakých bodech.*

- $f(x, y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$,
  $u(x, y) = y^3 - 3x^2y$,
  $v(x, y) = x^3 - 3xy^2 + 2$,
  $\frac{\partial u}{\partial x} = -6xy$, $\frac{\partial v}{\partial y} = -6xy$
  $\frac{\partial u}{\partial y} = 3y^2 - 3x^2$, $-\frac{\partial v}{\partial x} = -(3x^2 - 3y^2) = 3y^2 - 3x^2$,
  f je holomorfní $\forall z = x + iy$.
- $f(x, y) = \frac{1}{z} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$,
  $u(x, y) = \frac{x}{x^2+y^2}$,
  $v(x, y) = \frac{y}{x^2+y^2}$,
  $\frac{\partial u}{\partial x} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial v}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$
  $\frac{\partial u}{\partial y} = -x(x^2 + y^2)^{-2} \cdot 2y = -2xy(x^2 + y^2)^{-2}$,
  $-\frac{\partial v}{\partial x} = -y(x^2 + y^2)^{-2} \cdot 2x = -2xy(x^2 + y^2)^{-2}$,
  f je holomorfní na $\mathbf{C} \setminus 0$.

Úkol 7: *Buď f holomorfní funkce a $u(x, y) = 3x^2y - y^3$ její reálná část. Nalezněte imaginární část $v(x, y)$ funkce f.*

Řešení.
$\frac{\partial u}{\partial y} = 3x^2 - 3y^2$,
$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow v = \int 3y^2 - 3x^2 \mathrm{d}x = -x^3 + 3xy^2 + c$,
$v(x, y) = -x^3 + 3xy^2 + c$.

Úkol 8: *Najděte "branch points" a "branch cups" funkcí.*

- $\sqrt[3]{z}$
  $\sqrt[3]{z} = 0$
  $z = 0$
  $\omega(z) = \sqrt[3]{z} = (re^{i\phi})^{1/3} = r^{1/3} e^{(i\phi + i2\pi n)^{1/3}}$, $\phi \in [0; 2\pi)$,
  $\omega(z) = \sqrt[3]{r} e^{i\frac{\phi}{3}} e^{i\frac{2}{3}\pi n}$,
  pro $n = 0 \longrightarrow e^{i\frac{2}{3}\pi * 0} = 1$,
  pro $n = 1 \longrightarrow e^{i\frac{2}{3}\pi} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.
  - $\omega(z)$ pro $z = 0$ je multipled valued, bod $0 \in \mathbf{C}$ je *branch point.*
  Branch cut je interval $[0; \pi)$.

- $\sqrt[n]{z}$
  $\sqrt[n]{z} = 0$
  $z = 0$
  $\omega(z) = \sqrt[n]{z} = (re^{i\phi})^{1/n} = r^{1/n} e^{(i\phi + i2\pi k)^{1/n}}$, $\phi \in [0; 2\pi)$,
  $\omega(z) = \sqrt[n]{r} e^{i\frac{\phi}{n}} e^{i\frac{2}{n}\pi k}$,
  pro $k = 0 \longrightarrow e^{i\frac{2}{n}\pi * 0} = 1$,
  pro $k = 1 \longrightarrow e^{i\frac{2}{n}\pi} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.
  - $\omega(z)$ pro $z = 0$ je multipled valued, bod $0 \in \mathbf{C}$ je *branch point.*
  Branch cut je interval $[0; \pi)$.

- $\frac{1}{\sqrt{z-1}}$
  $|z - 1| = 0$, $|z| = 1$
  $z - 1 = re^{i\phi}$,
  $\omega(z) = \frac{1}{\sqrt{re^{i\phi}}} = (re^{i\phi})^{-1/2} = r^{-\frac{1}{2}} e^{-i\frac{\phi}{2}} e^{-i\frac{2\pi n}{2}}$, $\phi \in [0; 2\pi)$,
  pro $n = 0 \longrightarrow e^{-i\frac{2\pi * 0}{2}} = 1$,
  pro $n = 1 \longrightarrow e^{-i\pi} = -1$,
  pro $n = 2 \longrightarrow e^{-i2\pi} = 1$.
  - $\omega(z)$ pro $z = 1$ je multipled valued, bod $1 \in \mathbf{C}$ je *branch point.*
  Branch cut je interval $[1; \pi)$.

- $(z + 1 - 2i)^{1/4}$
  $z + 1 - 2i = 0$, $z = 2i - 1$
  $z + 1 - 2i = re^{i\phi}$,
  $\omega(z + 1 - 2i) = (re^{i\phi})^{1/4} = r^{\frac{1}{4}} e^{i\frac{\phi}{4}} e^{i\frac{2\pi}{4} n}$, $n \in \mathbf{Z}$, $\phi \in [0; 2\pi)$,
  pro $n = 0 \longrightarrow e^{i\frac{2\pi}{4} * 0} = 1$,

pro $n = 1 \longrightarrow e^{i\frac{2\pi}{4}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$,

pro $n = 2 \longrightarrow e^{i\pi} = 1$.

- $\omega(z)$ pro $z = 2i - 1$ je multipled valued, bod $(2i-1) \in \mathbf{C}$ je *branch point*.

Branch cut je interval $[2i - 1; \infty)$.

Úkol 9: *Spočtěte všechny hodnoty a "principal values".*

- $\log(1 + \sqrt{3}i)$

  $z = a + ib = 1 + i\sqrt{3}$, $r = |z| = \sqrt{4} = 2$,

  $\cos\phi = \frac{a}{|r|} = \frac{1}{2}$, $\sin\phi = \frac{\sqrt{3}}{2} \longrightarrow \phi = \frac{\pi}{3}$.

  $z = 1 + i\sqrt{3} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2e^{i\frac{\pi}{3}} = e^{\ln 2}e^{i\frac{\pi}{3}}$,

  $\omega = \log r = \log 2 + i\frac{\pi}{3} + i2n\pi = \log r + i(\frac{\pi}{3} + 2n\pi)$.

  - *Principal value:* $n = 0 \longrightarrow \omega = \log 2 + i\frac{\pi}{3}$,

  pro $n \in \mathbf{Z} \longrightarrow \omega = \log 2 + i(\frac{\pi}{3} + 2n\pi)$.

- $\log i^3$

  $z = i^3 = -i$, $r = |z| = 1$,

  $\cos\phi = \frac{0}{1} = 0$, $\sin\phi = \frac{-1}{1} \longrightarrow \phi = \frac{3\pi}{2}$.

  $\omega = \log r = \log 1 + i\frac{3\pi}{2} + 2ni\pi = \log 1 + i(\frac{3\pi}{2} + 2n\pi)$, $\forall n \in \mathbf{Z}$.

  - *Principal value:* $\omega = log 1 + i\frac{3\pi}{2} = i\frac{3\pi}{2}$.

Úkol 10: *Dokažte $z_n \to z$ právě tehdy, když $Re\, z_n \to Re\, z$ a zároveň $Im\, z_n \to Im\, z$.*

Řešení.

$\lim_{n\to\infty} z_n = y \longleftrightarrow \forall\epsilon > 0, \exists N \in \mathbf{N} : \forall n \in \mathbf{N}\ |z_n - y| < \epsilon$.

- $\Longrightarrow$: $\lim_{n\to\infty} z_n = \lim_{n\to\infty}(Re\ z_n + Im\ z_n) = \lim_{n\to\infty} Re\ z_n + i\lim_{n\to\infty} Im\ z_n = Re\ z + i \cdot Im\ z = z$,

- $\Longleftarrow$: $\lim_{n\to\infty} Re\ z_n = Re\ z$ & $\lim_{n\to\infty} Im\ z_n = Im\ z$,

  $\lim_{n\to\infty} Re\ z_n + i \cdot \lim_{n\to\infty} Im\ z_n = \lim_{n\to\infty}(Re\ z_n + i \cdot Im\ z_n) = Re\ z + i \cdot Im\ z = z$.

Úkol 11: *Dokažte, že posloupnost komplexních čísel $\{a_n\}$ je Cauchyovská právě tehdy, když jsou Cauchyovské posloupnosti realných čísel $\{Re\ a_n\}$ a $\{Im\ a_n\}$.*

Řešení.

- $\Longrightarrow$: $|a_m - a_n| < \epsilon$

  $|Re\ a_m + i \cdot Im\ a_m - (Re\ a_n + i \cdot Im\ a_n)| < \epsilon\ /^2$

  $|Re\ a_m - Re\ a_n + i(Im\ a_m - Im\ a_n)|^2 < \epsilon^2$

  $(Re\ a_m - Re\ a_n)^2 + (Im\ a_m - Im\ a_n)^2 < \epsilon^2$

  - součet 2 nezáporných reálných čísel menších než $\epsilon^2 \to$ každé z nich je tedy menší než $\epsilon^2 \to (Re\ a_m - Re\ a_n)^2 < \epsilon^2$, tedy $\{Re\ a_n\}$ je Cauchyovská a $(Im\ a_m - Im\ a_n)^2 < \epsilon^2$, tedy $\{Im\ a_n\}$ je Cauchyovská.

- $\Longleftarrow$: $|Re\ a_m - Re\ a_n| < \epsilon$ & $|Im\ a_m - Im\ a_n| < \epsilon$

  $|Re\ a_m - Re\ a_n| + |Im\ a_m - Im\ a_n| < 2\epsilon$

  $|Re\ a_m - Re\ a_n + Im\ a_m - Im\ a_n| \leq |Re\ a_m - Re\ a_n| + |Im\ a_m - Im\ a_n| < 2\epsilon$

  $|a_m - a_n| < 2\epsilon \to \{a_n\}$ je Cauchyovská.

Úkol 12: *Najděte rádius konvergence pro řady:*

- $\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$,

  $\frac{1}{R} = \limsup_{n\to\infty} |\frac{1}{(n+1)!}|^{1/n}$.

  Uvažujme celé číslo M, pro $(n+1) \geq M$ dostáváme: $(n+1)! = M!(M+1)(M+2)...(n)(n+1) \geq M!M^{(n+1)-M}$,

  $[(n+1)!]^{1/n} \geq (M!)^{1/n} M^{1+\frac{1-M}{n}} = (M!)^{1/n} M^{1-\frac{M-1}{n}}$,

  pro $n \to \infty$: $(M!)^0 M^{1-0}$, tedy $[(n+1)!]^{1/n}$ bude pro dostatečně velké $n \geq \frac{M}{2}$, M bylo libovolné $[(n+1)!]^{1/n} \to \infty$, poloměr konvergence $R = \infty$.

- $\sum_{n=0}^{\infty} n^n z^n$,

  $\frac{1}{R} = \limsup_{n\to\infty} |n^n|^{1/n} = \limsup_{n\to\infty} |n^{n/n}| = \limsup_{n\to\infty} |n| = \infty$,

  $R = 0 \to$ řada konverguje pouze v $z = 0$.

- $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$,

  Analogicky jako v prvním příkladu: $[(2n)!]^{1/n} \geq (M!)^{1/n} M^{\frac{2n-M}{n}} = (M!)^{1/n} M^{2-\frac{M}{n}} \to$ $R = \infty$

Úkol 13: *Najděte rádius konvergence pro řady:*

- $\sum_{n=0}^{\infty} z^{n!}$,
  $\sum_{n=0}^{\infty} z^{n!} = z + z + z^2 + z^6 + z^{24} + ...$
  $\sum_{n=0}^{\infty} z^{n} = 1 + z + z^2 + z^3 + z^4 + ...$ - geometrická řada lim $\frac{1}{1-z}$ existuje pro $|n| < 1$,
  $\sum_{n=0}^{\infty} z^{n!}$ je vybraná podposloupnost částečných součtů z geometrické řady, její limita tedy existuje také pro $|z| < 1$ a poloměr konvergence $R = 0$.
- $\sum_{n=0}^{\infty} q^{n^2} z^n, q \in (0,1)$,
  $\limsup_{n\to\infty} |q^{n^2}|^{1/n} = \limsup_{n\to\infty} |q^n| = 0 \to R = \infty$.

Úkol 14: *Spočtětě sumu* $\sum_{n=0}^{\infty}(n+1)^2 z^n, |z| < 1$.

Řešení.
Víme, že $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ pro $|z| < 1$,
$(\sum_{n=0}^{\infty} z^n)' = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n$
$(\frac{1}{1-z})' = \frac{(1-z)-1}{(1-z)^2} = \frac{-z}{(1-z)^2}$
$\to \sum_{n=1}^{\infty} nz^{n-1} = \frac{-z}{(1-z)^2}$,
$(\sum_{n=1}^{\infty} nz^{n-1})' = \sum_{n=2}^{\infty} n(n-1)z^{n-2} = \sum_{n=0}^{\infty}(n+1)^2 z^n$
$(\frac{-z}{(1-z)^2})' = \frac{(-1)(1-z)^2+2z(1-z)}{(1-z)^4} = \frac{-1+4z-z^2}{(1-z)^4}$
$\to \sum_{n=0}^{\infty}(n+1)^2 z^n = \frac{-1+4z-z^2}{(1-z)^4}$.

Úkol 15: *Ukažte, že* $\int_{\phi+\psi} f = \int_\phi f + \int_\psi f$, $\int_{-\phi} f = -\int_\phi f$.
Řešení.

- $\int_{\phi+\psi} f = \int_\phi f + \int_\psi f$,
  $\int_\phi f = \int_a^b f(\phi(t))\phi'(t)\,dt$
  $\int_\psi f = \int_c^d f(\psi(t))\psi'(t)\,dt$
  $\to \int_\phi f + \int_\psi f = \int_a^b f(\phi(t))\phi'(t)\,dt + \int_c^d f(\psi(t))\psi'(t)\,dt = \int_{a+c}^{b+d} f(\phi(t))\phi'(t) + f(\psi(t))\psi'(t)\,dt = \int_{\phi+\psi} f$.
- $\int_{-\phi} f = -\int_\phi f$,
  $\to \int_{-\phi} f = \int_b^a f(\phi(t))\phi'(t)\,dt = -\int_a^b f(\phi(t))\phi'(t)\,dt = -\int_\phi f$.

Úkol 16: *Spočtětě:*

- $\int_\gamma |z|\overline{z}\mathrm{d}z$, kde $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, γ_1 je úsečka z bodu 0 do bodu 1, γ_2 je výsek jednotkové kružnice z bodu 1 do bodu $e^{i\pi/4}$ a γ_3 je úsečka z bodu $e^{i\pi/4}$ do bodu 0.
  - $\int_{\gamma_1} |z|\overline{z}\mathrm{d}z = |\gamma_1 : 0 \to 1, z = \gamma_1(t), \gamma_1(t) = t, t \in [0,1]| = \int_0^1 \sqrt{t^2}t\mathrm{d}t = [\frac{t^3}{3}]_0^1 = \frac{1}{3}$,
  - $\int_{\gamma_2} |z|\overline{z}\mathrm{d}z = |\gamma_2 : 1 \to e^{\frac{i\pi}{4}}, z = e^{i\phi}, \phi \in [0, \frac{\pi}{4}]| = \int_0^{\frac{\pi}{4}} |e^{i\phi}|ie^{i\phi}e^{-i\phi}\mathrm{d}\phi = \int_0^{\frac{\pi}{4}} |e^{i\phi}|i\mathrm{d}\phi = i\int_0^{\frac{\pi}{4}} |\cos\phi + i\sin\phi|\mathrm{d}\phi = i\int_0^{\frac{\pi}{4}} \mathrm{d}\phi = i\frac{\pi}{4}$,
  - $\int_{\gamma_3} |z|\overline{z}\mathrm{d}z = |\gamma_3 : e^{i\frac{\pi}{4}} \to 0, z = \gamma_3(t), \gamma_3(t) = (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) + (0 - \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})t, t \in [0,1]| = \int_0^1 \sqrt{(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^2 + (-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})^2t^2}[(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) + (-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})t](-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})\mathrm{d}t = \int_0^1 \sqrt{(\frac{2}{4} + \frac{2i}{2} - \frac{2}{4}) + (\frac{2}{4} + \frac{2i}{2} - \frac{2}{4})t^2}[\frac{2}{4} + \frac{2}{4} + t](-1)\mathrm{d}t = \int_0^1 \sqrt{i - it^2}(-t)\mathrm{d}t = |i - it^2 = s| = \int_\alpha^\beta \sqrt{s}\frac{-\mathrm{d}s}{-2i} = -\frac{1}{2i}\int_\alpha^\beta s^{\frac{1}{z}}\mathrm{d}s = -\frac{1}{2i}[\frac{2s^{(\frac{3}{2}}}{3}]_\alpha^\beta = -\frac{1}{2i}[\frac{2}{3}(i - it^2)^{\frac{3}{2}}]_0^1 = -\frac{1}{2i}[\frac{2}{3}(i - i)^{\frac{3}{2}} - \frac{2}{3}(i - i0)^{\frac{3}{2}}] = -\frac{1}{2i}(-\frac{2}{3}i^{\frac{3}{2}}) = \frac{1}{3}\sqrt{i}$,

  Celkově: $\int_\gamma |z|\overline{z}\mathrm{d}z = \frac{1}{3} + i\frac{\pi}{4} + \sqrt{i}\frac{1}{3}$.

- $\int_\gamma e^z\mathrm{d}z$, kde γ je výsek hyperboly $(Re\ z)^2 - (Im\ z)^2 = 1$ z bodu $\sqrt{2} - i$ do bodu $\sqrt{2} + i$.

  \- Řešení: obrázek,

- $\int_\gamma ze^z\mathrm{d}z$, kde γ je čtverec z bodů $\{0, 1, 1+i, i\}$.

  $\to f = ze^z$ je holomorfní - součin 2 holomorfních funkcí, γ je jednoduchá uzavřená křivka $\to$ Věta: $\int_\gamma f = 0$.

- $\int_\gamma e^z(z^{12} + 2)^{-1} - 2\overline{z}\mathrm{d}z$, kde γ je jednotková kružnice.

  $\to \int_\gamma \frac{e^z}{z^{12}+2}\mathrm{d}z - \int_\gamma 2\overline{z}\mathrm{d}z = \int_\gamma \frac{1+z+\frac{z^2}{2!}+\ldots}{z^{12}+2}\mathrm{d}z - \int_\gamma 2\overline{z}\mathrm{d}z = 0 - \int_\gamma 2\overline{z}\mathrm{d}z = |z = e^{i\phi}, dz, \phi \in [0, 2\pi]| = \int_0^{2\pi} 2e^{-i\phi}ie^{i\phi}\mathrm{d}\phi = 2i[2\pi - 0] = -4\pi i$.

  Poznámka: 0 - singularita mimo γ - Věta jako v (c).

Úkol 17: *Stanovte všechny izolované singulární body následujících funkcí a určete jejich typy:*

- $\frac{e^{1/(z-1)}}{e^z-1}$,
  - $e^z - 1 \neq 0$
    $e^z \neq 1$
    $z \neq 0$
    $\lim_{z\to 0} \frac{e^{1/(z-1)}}{e^z-1} \to$ limita neexistuje, protože: $\lim_{z\to 0^+} = \infty$ a $\lim_{z\to 0^-} = -\infty$
    $\to$ pro $z = 0$ je singularita esenciální,
  - $z - 1 \neq 0$
    $z = 1$
    $\lim_{z\to 1} \frac{e^{1/(z-1)}}{e^z-1} \to$ limita neexistuje, protože: $\lim_{z\to 1^+} = \infty$ a $\lim_{z\to 1^-} = 0$
    $\to$ pro $z = 1$ je singularita esenciální,
- $\frac{1}{\sin(1/z)}$,
  - $\sin\frac{1}{z} = 0 \to \frac{1}{z} = k\pi \to z = \frac{1}{k\pi}, k \in \mathbf{Z} \setminus \{0\}$
    $\lim_{z\to\frac{1}{k\pi}} \frac{1}{\sin(1/z)} \to$ limita neexistuje, protože: $\lim_{z\to\frac{1}{k\pi}^+} = \infty$ a $\lim_{z\to\frac{1}{k\pi}^-} = -\infty$
    $\to$ v bodech $z = \frac{1}{k\pi}, k \in \mathbf{Z} \setminus \{0\}$ je esenciální singularita.

Úkol 18: *Nalezněte rozvoj funkce $f(z) = z\sin(1/z)$ v Laurentovu řadu se středy v bodech 0 a ∞. Dále stanovte koeficienty a_{-2}, a_{10}.*

Řešení.

- se středem v $z = 0$
  $f(z) = z\sin\frac{1}{z} \to$ nelze,
- se středem v $z = \infty$
  $f(z) = z\sin\frac{1}{z} \to \sin\frac{1}{z} = \frac{e^{i\frac{1}{z}} - e^{i\frac{1}{z}}}{2i}$
  $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
  $e^{i\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{(i\frac{1}{z})^n}{n!}$
  $e^{-i\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{(-i\frac{1}{z})^n}{n!}$
  $f(z) = z(\frac{1}{2i}\sum_{n=0}^{\infty} \frac{(i\frac{1}{z})^n}{n!} - \frac{1}{2i}\sum_{n=0}^{\infty} \frac{(-i\frac{1}{z})^n}{n!}) = z\frac{1}{2i}\sum_{n=0}^{\infty} \frac{(i\frac{1}{z})^n(-i\frac{1}{z})^n}{n!} = \frac{z}{2i}\sum_{n=0}^{\infty} \frac{z^{-n}[i^n-(-i)^n]}{n!} = \frac{1}{2}\sum_{n=-1}^{\infty} \frac{z^{-n+1}[i^{n-1}-(-i)^{n-1}]}{n!}$.

$a_{-2} = 0$,
$a_{10} = \frac{1}{2}\frac{i^{10-1}-(-i)^{10-1}}{10!} = \frac{1}{2}\frac{i-(-i)}{10!} = \frac{i}{10!}$.

Úkol 19: *Dokažte, že singularita je pólem právě tehdy, když jenom konečně mnoho členů Laurentova rozvoje $a_n, n < 0$ je nenulových.*

Řešení.

Singularita je pólem právě tehdy, když pouze konečně mnoho členů Laurentova rozvoje $a_n,\ n < 0$ je nenulových.

Úkol 20: *U následujících funkcí nalezněte body, v nichž má funkce singularity. Dále určete jejich typ (včetně násobnosti, pokud se jedná o pól):*

- $\frac{z^3}{z^2+z+1}$,

  $z^2 + z + 1 = 0 \rightarrow z_{1,2} = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}i}{2}$,

  $\lim_{z\rightarrow\frac{-1+\sqrt{3}i}{2}} \frac{z^3}{z^2+z+1} = \infty$

  $\lim_{z\rightarrow\frac{-1+\sqrt{3}i}{2}} (z-\frac{-1+\sqrt{3}i}{2})(\frac{z^3}{(z-\frac{-1-\sqrt{3}i}{2})(z-\frac{-1+\sqrt{3}i}{2})}) = \lim_{z\rightarrow\frac{-1+\sqrt{3}i}{2}} \frac{z^3}{(z-\frac{-1-\sqrt{3}i}{2})} = \frac{(\frac{-1-\sqrt{3}i}{2})(\frac{-1+\sqrt{3}}{2})}{(\frac{-1+\sqrt{3}i}{2})-(\frac{-1-\sqrt{3}i}{2})} = \frac{\frac{3}{4}-\frac{1}{4}}{\sqrt{3}i} = \frac{1}{2\sqrt{3}i} \rightarrow$ pól 1. řádu

  $\lim_{z\rightarrow\frac{-1-\sqrt{3}i}{2}} (z-\frac{-1-\sqrt{3}i}{2})\frac{z^3}{(z-\frac{-1-\sqrt{3}i}{2})(z-\frac{-1+\sqrt{3}i}{2})} = \lim_{z\rightarrow\frac{-1-\sqrt{3}i}{2}} \frac{z^3}{z-\frac{-1+\sqrt{3}i}{2}} = \frac{(\frac{-1+\sqrt{3}i}{2})(\frac{-1-\sqrt{3}i}{2})}{\frac{-1-\sqrt{3}i}{2}-(\frac{-1+\sqrt{3}i}{2})} = \frac{\frac{1}{4}+\frac{3}{4}}{-\sqrt{3}i} = -\frac{1}{\sqrt{3}i} \rightarrow$ pól 1. řádu

- $\frac{e^{z^2}-1}{z^2}$,

  $z = 0$

  $\lim_{z\rightarrow 0} \frac{e^{z^2}-1}{z^2} = \frac{e^{z^2}2z}{2z} = 1 \rightarrow$ odstranitelná singularita

- $\frac{e^{2z}-1}{z^2}$,

  $z = 0$

  $\lim_{z\rightarrow 0} \frac{e^{2z}-1}{z^2} = \lim_{z\rightarrow 0} \frac{e^{2z}2}{2z} = \infty$

  $\lim_{z\rightarrow 0} z\frac{e^{2z}-1}{z^2} = \lim_{z\rightarrow 0} \frac{e^{2z}-1}{z} = \lim_{z\rightarrow 0} \frac{e^{2z}2}{1} = 2 \rightarrow$ pól 1. řádu

Úkol 21: *Spočtěte* $\oint_{C=1} f(z)\mathrm{d}z$ *následujících funkcí:*

- $\frac{z+1}{2z^3-3z^2-2z}$,

  $2z^3 - 3z^2 - 2z = 0$

  $z(2z^2 - 3z - 2) = 0$

  $z_1 = 0,\ z_{2,3} = \frac{3\pm\sqrt{9-4\cdot 2\cdot(-2)}}{4} = \frac{3\pm\sqrt{25}}{4} \to z_2 = 2,\ z_3 = -\frac{1}{2}$

  $2z^3 - 3z^2 - 2z = 2z(z-2)(z+\frac{1}{2})$

  Singularity: uvnitř $c = 1 \to z_1 = 0,\ z_2 = -\frac{1}{2}$, mimo $c = 1 \to z_3 = 2$.

  Pomocí Reziduové věty: $\oint_\phi f(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{m} Res_{z_j} f\ Ind_\phi z_j,\ c = 1 \to Ind_\phi z_j = 1$

  $\oint_{c=1} \frac{z+1}{2z^3-3z^2-2z} = 2\pi i(Res_{z=0} + Res_{z=-\frac{1}{2}})$

  $Res_{z=0} \frac{z+1}{2z(z-2)(z+\frac{1}{2})} \overset{(1)}{=} \lim_{z\to 0} z \frac{z+1}{2z(z-2)(z+\frac{1}{2})} = \frac{1}{2(-2)(\frac{1}{2})} = -\frac{1}{2}, \to (1):$ $z = 0$ je pól 1. řádu.

  $Res_{z=-\frac{1}{2}} \frac{z+1}{2z(z-2)(z+\frac{1}{2})} \overset{(2)}{=} \lim_{(z\to-\frac{1}{2}} (z+\frac{1}{2}) \frac{z+1}{2z(z-2)(z+\frac{1}{2})} = \frac{\frac{1}{2}}{\frac{5}{2}} = \frac{1}{5}, \to (2):$ $z = -\frac{1}{2}$ je pól 1. řádu.

  $\to \oint_{c=1} \frac{z+1}{2z(z-2)(z+\frac{1}{2})}\mathrm{d}z = 2\pi i(-\frac{1}{2} + \frac{1}{5}) = -\frac{3\pi i}{5}$,

- $\frac{\cosh(1/z)}{z}$,

  $\cosh(1/z) = \frac{e^{1/z}+e^{-1/z}}{2} \to \frac{\cosh(1/z)}{z} = \frac{e^{1/z}+e^{-1/z}}{2z}$,

  $e^{1/z}$ má v $z = 0$ esenciální singularitu

  $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n} \cdot \frac{1}{n!} = \sum_{n=-\infty}^{-1} + \sum_{n=0}^{\infty} \frac{1}{z^n} \cdot \frac{1}{n!}$

  $e^{-1/z} = \sum_{n=0}^{\infty} (-\frac{1}{z^n}) \cdot \frac{1}{n!} = \sum_{n=-\infty}^{-1} + \sum_{n=0}^{\infty} \frac{1}{z^n} \cdot \frac{1}{n!}$

  $\frac{1}{z} = \sum_{n=-\infty}^{-2} 0 + 1z^{-1} + \sum_{n=0}^{\infty} 0$

  $\to Res_{z=0} \frac{1}{z} = 1,\ Res_{z=0} e^{\frac{1}{z}} = 0,\ Res_{z=0} e^{-\frac{1}{z}} = 0$

  $Res_{z=0}(\frac{e^{1/z}}{2z} + \frac{e^{-1/z}}{2z}) = Res_{z=0} \frac{e^{1/z}}{2z} + Res_{z=0} \frac{e^{-1/z}}{2z} = 0 + 0 = 0$

  $\to \oint_{c=1} \frac{\cosh(1/z)}{z} = 2\pi i \cdot 0 = 0$

- $\frac{e^{-\cosh z}}{4z^2+\pi^2}$

  $4z^2 + \pi^2 = 0$

  $z^2 = -\frac{\pi^2}{4}$

  $z = \pm\sqrt{-\frac{\pi^2}{4}} = \pm\frac{\pi}{2} i$

  $Res_{z=\frac{i\pi}{2}} = \frac{e^{-\cosh z}}{4z^2+\pi^2} = \lim_{z\to\frac{i\pi}{2}} (z - \frac{i\pi}{2}) \frac{e^{-\cosh z}}{4z^2+\pi^2} = \frac{e^{-\cosh z}}{4(\frac{i\pi}{2}+\frac{i\pi}{2})} = \frac{1}{4i\pi} = -\frac{i}{4\pi}$

  $Res_{z=-\frac{i\pi}{2}} = \frac{e^{-\cosh z}}{4z^2+\pi^2} = \lim_{z\to-\frac{i\pi}{2}} (z + \frac{i\pi}{2}) \frac{e^{-\cosh z}}{4(\frac{i\pi}{2}+\frac{i\pi}{2})} = \frac{1}{4(-\frac{i\pi}{2}-\frac{i\pi}{2})} = \frac{1}{-4i\pi} = \frac{i}{4\pi}$,

$\rightarrow \oint_{c=1} \frac{e^{-\cosh z}}{4z^2+\pi^2}\mathrm{d}z = 2\pi i(-\frac{i}{4\pi} + \frac{i}{4\pi}) = 0.$

Úkol 22: *Spočtěte z definice Laplaceovy transformace funkcí:*

- $te^{at} \rightarrow F(p) = \int_0^\infty e^{-tp} f(t)\mathrm{d}t$,

  $F(p) = \int_0^\infty e^{-tp} te^{at}\mathrm{d}t = \int_0^\infty e^{at-tp}\mathrm{d}t = \int_0^\infty e^{t(a-p)}\mathrm{d}t \overset{perpartes}{=} [\frac{t}{a-p}e^{t(a-p)}]_0^\infty - \int_0^\infty \frac{1}{a-p}e^{t(a-p)}\mathrm{d}t = [\frac{t}{a-p}e^{t(a-p)}]_0^\infty - [\frac{1}{(a-p)^2}e^{t(a-p)}]_0^\infty = [\frac{t}{a-p}e^{t(a-p)}]_0^\infty - [\frac{1}{(a-p)^2}e^{-t(p-a)}]_0^\infty = 0 - [0 - \frac{1}{(a-p)^2} = \frac{1}{(a-p)^2}.$

- $\sin(at)$,

  $\int e^{-tp}\sin(at)\mathrm{d}t \overset{perpartes}{=} \frac{\sin(at)}{-p}e^{-tp} + \frac{a}{p}\int \cos(at)e^{-tp}\mathrm{d}t \overset{perpartes}{=} \frac{\sin(at)}{-p}e^{-tp} + \frac{a}{p}[\frac{\cos(at)}{-p}e^{-tp} - \frac{a}{p}\int \sin(at)e^{-tp}\mathrm{d}t] = \frac{\sin(at)}{-p}e^{-tp} + \frac{a}{p^2}\cos(at)e^{-pt} - \frac{a^2}{p^2}\int \sin(at)e^{-tp}\mathrm{d}t$

  $\rightarrow (1 + \frac{a^2}{p^2})\int e^{-tp}\sin(at)\mathrm{d}t = \frac{\sin(at)}{-p}e^{-tp} - \frac{a}{p^2}\cos(at)e^{-tp}$

  $\int e^{-tp}\sin(at)\mathrm{d}t = \frac{p^2}{p^2+a^2}\frac{\sin(at)}{-p}e^{-tp} - \frac{p^2}{p^2+a^2}\frac{a}{p^2}\cos(at)e^{-tp}$

  $\int e^{-tp}\sin(at)\mathrm{d}t = -\frac{p}{p^2+a^2}\sin(at)e^{-tp} - \frac{a}{p^2+a^2}\cos(at)e^{-tp}$

  $F(p) = \int_0^\infty e^{-tp}\sin(at)\mathrm{d}t = [-\frac{p}{p^2+a^2}]\sin(at)e^{-tp} - \frac{a}{p^2+a^2}\cos(at)e^{-tp}]_0^\infty = (-\frac{p}{p^2+a^2}\sin(at)\cdot 0 - \frac{a}{p^2+a^2}\cos(at)\cdot 0) - (-\frac{p}{p^2+a^2}\sin 0 - \frac{a}{p^2+a^2}\cos 0) = \frac{a}{p^2+a^2}$

- $\cos(at)$,

  $\int e^{-tp}\cos(at)\mathrm{d}t \overset{perpartes}{=} \frac{\cos(at)}{-p}e^{-tp} - \frac{a}{p}\int \sin(at)e^{-tp}\mathrm{d}t \overset{perpartes}{=} \frac{\cos(at)}{-p}e^{-tp} - \frac{a}{p}[e^{-tp}\frac{\sin(at)}{-p} + \frac{a}{p}\int \cos(at)e^{-tp}\mathrm{d}t] = \frac{cos(at)}{-p}e^{-tp} + \frac{a}{p^2}e^{-tp}\sin(at) - \frac{a^2}{p^2}\int \cos(at)e^{-tp}\mathrm{d}t$

  $\rightarrow (1 + \frac{a^2}{p^2})\int e^{-tp}\cos(at)\mathrm{d}t = \frac{cos(at)}{-p}e^{-tp} + \frac{a}{p^2}e^{-tp}\sin(at)$

  $\int e^{-tp}\cos(at)\mathrm{d}t = \frac{p^2}{p^2+a^2}\frac{\cos(at)}{-p}e^{-tp} + \frac{p^2}{p^2+a^2}\frac{a}{p^2}\sin(at)e^{-tp}$

  $\int e^{-tp}\cos(at)\mathrm{d}t = -\frac{p}{p^2+a^2}\cos(at)e^{-tp} + \frac{a}{p^2+a^2}\sin(at)e^{-tp}$

  $F(p) = \int_0^\int e^{-tp}\cos(at)\mathrm{d}t = [-\frac{p}{p^2+a^2}\cos(at)e^{-tp} + \frac{a}{p^2+a^2}\sin(at)e^{-tp}]_0^\infty = -(-\frac{p}{p^2+a^2}\cos 0 + \frac{a}{p^2+a^2}\sin 0) = \frac{p}{p^2+a^2}$

Úkol 23: *Spočtěte:*

- $\int_0^\infty \frac{\mathrm{d}x}{x^6+1}$,

  $\frac{1}{x^6+1}$ je sudá funkce $\to \int_0^\infty \frac{1}{x^6+1}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{x^6+1}\mathrm{d}x$

  $(z^6+1) = (z^3+i)(z^3-i)$

  $z_1 = (e^{i\frac{3\pi}{2}})^{1/3} = i \to z_2 = -i$

  $z_3 = (e^{i\frac{\pi}{2}})^{1/3} = \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + \frac{i}{2} \to z_4 = \frac{\sqrt{3}}{2} - \frac{i}{2}$

  $z_5 = (e^{i\frac{\pi}{2}})^{5/3} = \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}) = \frac{-\sqrt{3}}{2} + \frac{i}{2} \to z_6 = \frac{-\sqrt{3}}{2} + \frac{i}{2}$

  Věta: Nechť f je holomorfní funkce na $T = \{z \in \mathbf{C}, Im\ z > 0\}$ a spojitá na uzávěru T až na konečný počet singularit mimo reálnou osu. Pokud pro $z \in \overline{T}$ je $\lim_{z\to\infty} zf(z) = 0$ a integrál $\int_{-\infty}^\infty f(x)\mathrm{d}x$ konverguje, platí $\int_{-\infty}^\infty f(x)\mathrm{d}x = 2\pi i \sum Res f(z)$.

  Kladnou imaginární část mají z_1, z_3, z_5:

  - $\lim_{z\to z_1}(z-z_1)\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)} = \frac{1}{(i-(-i))(i-\frac{\sqrt{3}+i}{2})(i-\frac{\sqrt{3}-i}{2})(i-\frac{-\sqrt{3}+i}{2})(i-\frac{-\sqrt{3}-i}{2})} = \frac{1}{2i(\frac{i-\sqrt{3}}{2})(\frac{3i-\sqrt{3}}{2})(\frac{i+\sqrt{3}}{2})(\frac{3i+\sqrt{3}}{2})} = \frac{8}{(i^2-\sqrt{3}i)(3i-\sqrt{3})(i+\sqrt{3})(3i+\sqrt{3})} = \frac{8}{(-1-\sqrt{3}i)(-9-3)(i+\sqrt{3})} = \frac{8}{12i+12\sqrt{3}+12\sqrt{3}i^2+12\cdot 3i} = \frac{8}{48i} = -\frac{i}{6}$

    $\to Res_{z=i}\frac{1}{z^6+1} = -\frac{i}{6}$
  - $\lim_{z\to z_3} \frac{1}{(\frac{\sqrt{3}+i}{2}-i)(\frac{\sqrt{3}+i}{2}+i)(\frac{\sqrt{3}+i-\sqrt{3}+i}{2})(\frac{\sqrt{3}+i-\sqrt{3}+i}{2})(\frac{\sqrt{3}+i+\sqrt{3}-i}{2})(\frac{\sqrt{3}+i+\sqrt{3}+i}{2})} = \frac{1}{(\frac{\sqrt{3}-i}{2})(\frac{\sqrt{3}+3i}{2})i\sqrt{3}(\frac{2\sqrt{3}+2i}{2})} = \frac{4}{(\sqrt{3}-i)(\sqrt{3}+3i)(\sqrt{3}i)(\sqrt{3}+i)} = \frac{4}{(3+1)(3i-3\sqrt{3})} = \frac{1}{(3i-3\sqrt{3})}$

    $\to Res_{z=\frac{\sqrt{3}+i}{2}}\frac{1}{z^6+1} = \frac{1}{(3i-3\sqrt{3})}$
  - $\lim_{z\to z_5} \frac{1}{(\frac{-\sqrt{3}+i}{2}-i)(\frac{-\sqrt{3}+i}{2}+i)(\frac{-\sqrt{3}+i-\sqrt{3}-i}{2})(\frac{-\sqrt{3}+i-\sqrt{3}+i}{2})(\frac{-\sqrt{3}+i+\sqrt{3}+i}{2})} = \frac{1}{(\frac{-\sqrt{3}-i}{2})(\frac{-\sqrt{3}+3i}{2})(\frac{-2\sqrt{3}}{2})(\frac{-2\sqrt{3}+2i}{2}} \frac{1}{3\sqrt{3}+3i}$

    $\to Res_{z=\frac{-\sqrt{3}+i}{2}}\frac{1}{z^6+1} = \frac{1}{(3i+3\sqrt{3})}$

  $\int_{-\infty}^\infty \frac{x^2-1}{(x^2+1)^2}\mathrm{d}x = \pi i(\frac{1}{(3i+3\sqrt{3})} + \frac{1}{(3i-3\sqrt{3})} - \frac{i}{6}) = \pi i(\frac{3\sqrt{3}-3i}{36} + \frac{-3\sqrt{3}-3i}{36} - \frac{i}{6}) = \pi i(-\frac{i}{6} - \frac{i}{6}) = \pi i\frac{-2i}{6} = \frac{\pi}{3}$.

- $\int_{-\infty}^\infty \frac{x^2-1}{(x^2+1)^2}\mathrm{d}x$

  $(x^2+1)^2 = (x-i)^2(x+i)^2$

  $Res_{z=i}\frac{x^2-1}{(x^2+1)^2} = \lim_{z\to i}[(z-i)^2\frac{z^2-1}{(z+i)^2(z-i)^2}]' = \lim_{z\to i}(\frac{z^2-1}{(z+i)^2})' = \lim_{z\to i}\frac{2z(z+i)^2-(z^2-1)2(z+i)}{(z+i)^4} = \frac{2i(i+i)-(-1-1)2}{2^3i^3} = \frac{4i^2+4}{2^3i^3} = 0$

  $\to \int_{-\infty}^\infty \frac{x^2-1}{(x^2+1)^2}\mathrm{d}x = 2\pi i(Res_{z=i}\frac{x^2-1}{(x^2+1)^2}) = 0.$