Dynamické systémy

kapitola 1.4 - Hyperbolicity

3. Suppose f is diffeomorphism. Prove that all hyperbolic periodic points are isolated.
   diffeomorphism - homeomorphism + spojité derivace
   Bijektivní $\Rightarrow$ hladké - ze spojitosti derivace (body vedle sebe nemůžou mít stejnou derivaci)

4. Show via an example that hyperbolic points need not be isolated. (hledám interval hyperbolických periodických bodů - obrázek kreslen na tabuli)

5. Find an example of a C^1 diffeomorphism with a non-hyperbolic fixed point which is an accumulation of other hyperbolic points.
   (zmenšující se vlnka, která se plazí po diagonále a blíží se k 1)

6. Discuss the dynamics of the family $f_\alpha(x) = x^3 - \alpha x$ for $-\infty < \alpha \leq 1$. Find all parameter values where bifurcations occur. Describe how the phase portrait of f_α changes at these points. (jenom pokud je zajímavé)

$$f_\alpha(x) = x^3 - \alpha x = x$$

$$x^3 - \alpha x - x = 0$$

$$x(x^2 - \alpha - 1) = 0$$

$$x = 0, \quad x^2 - \alpha - 1 = 0$$

$$x^2 = \alpha + 1 \Rightarrow x = \pm\sqrt{\alpha + 1}$$

$$f_\alpha(f_\alpha(x)) = x$$

$$(x^3 - \alpha x)^3 - \alpha(x^3 - \alpha x) = x$$

kapitola 1.5 - Quadratic family

1. Prove that $F_2(x) = 2x(1-x)$ satisties: if $0 < x < 1$ then $F_2^n(x) \to \dfrac{1}{2}$ as $n \to \infty$
   Fixed points:

$$2x(1-x) = x$$

$$2x - 2x^2 = x$$

$$-2x^2 + x = 0$$

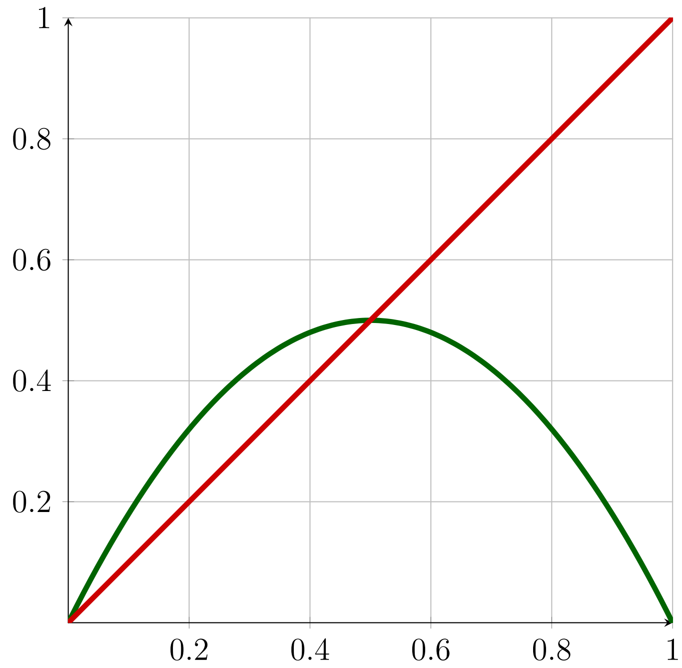
$$x(2-x) = 0$$

$$x = 0, x = \frac{1}{2}$$

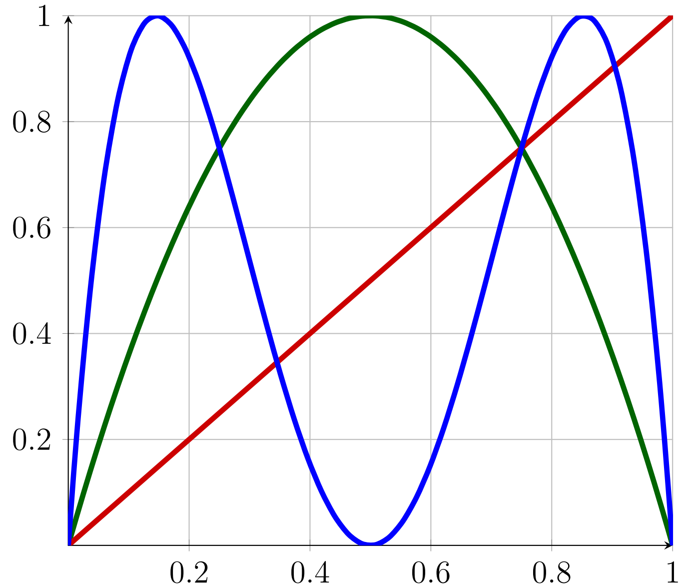
Hyperbolicity:

$$F_2'(x) = (2x - 2x^2)' = 2 - 4x$$

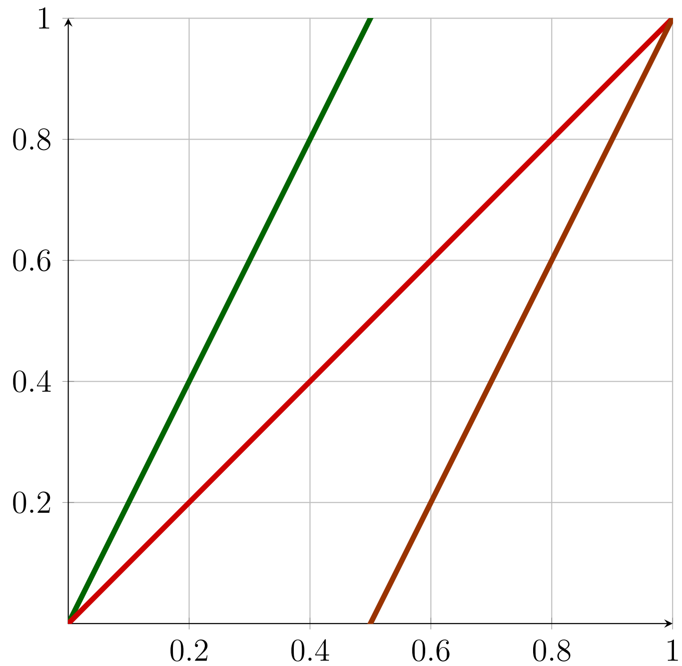
$F_2'(0) = 2 - 4 \cdot 0 = 2 > 1 \Rightarrow 0$ is repelling
$F_2'(\frac{1}{2}) = 2 - 4 \cdot \frac{1}{2} = 2 - 2 = 0 \Rightarrow \frac{1}{2}$ is attracting



2. Sketch the graph of $F_4(x) = 4x(1 - x)$ on the unit interval.



4. Prove that the set of all periodic points of $T(x)$ are dense in [0,1].
5. Baker map



no periodic points

6. $F(x) = x^3 - \lambda x$

a. Find all periodic pts and clasify them for $0 < \lambda < 1$
Fixed point:

$$x^3 - \lambda x = x$$
$$x^3 - x(\lambda + 1) = 0$$
$$x(x^2 - \lambda - 1) = 0$$
$$x = 0, x^2 = \lambda + 1$$
$$x = 0, x = \pm\sqrt{\lambda + 1}$$

Hyperbolicity:

$$F'(x) = 3x^2 - \lambda$$

$F'(0) = -\lambda \Rightarrow$ attracting for $|\lambda| < 1$, repelling for $|\lambda| > 1$
$F'(\sqrt{\lambda + 1}) = 3 \cdot (\lambda + 1) - \lambda = 3\lambda + 3 - \lambda = 2\lambda + 3 \Rightarrow$
$F'(\sqrt{\lambda + 1}) = 3 \cdot (-\lambda - 1) - \lambda = -4\lambda - 3 \Rightarrow$

b.

c.

7. Prove that the Cantor Middle-Thirds set is closed, nonempty, perfect, and totally disconnected.
perfect - doesn't have isolated points

nonempty: points on the edge - $0, \frac{1}{3}, \frac{2}{3}, 1 \in C \rightarrow$ *nonempty* (odebírám otevřené intervaly)

closed: doplněk je spočetné sjednocení otevřených ontervalů - otevřená množina

perfect: 1. iterace - vždy je bod ve vzdálenosti max $\frac{1}{3}$, 2. iterace - vždy je bod ve vzdálenosti max $\frac{1}{9}$...

totally disconected: z konstrukce

10. Let Γ be the Cantor Middle-Thirds set. Prove that the linear map $L(x) = 3x$ maps $\Gamma \cap [0, \frac{1}{3}]$ homeomorphically onto Γ.
Homeomorphism:

1. bijection: linear
2. continuous: linear function

3. continuous inverse: $\frac{1}{3}x$ - linear

11. Generalize ex. 10 - interval remaining at nth stage - $[0, \frac{1}{3^n}]$
$L_n(x) = 3^n x$
Homeomorphism:

    1. bijection: linear
    2. continuous: linear function
    3. continuous inverse: $\frac{1}{3^n}x$ - linear

kapitola 1.6 - Symbolic dynamics

1. Let $s = (001001001...), t = (010101...), r = (101010...)$. Compute:

    1. $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \frac{|0-0|}{1} + \frac{|0-1|}{2} + \frac{|1-0|}{2^2} + \frac{|0-1|}{2^3} + \frac{|0-0|}{2^4} + ... = 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{0}{16} + ...$
    2. $d[t,r] = \sum_{i=0}^{\infty} \frac{|t_i - r_i|}{2^i} = \frac{|0-1|}{1} + \frac{|1-0|}{2} + \frac{|0-1|}{2^2} + \frac{|1-0|}{2^3} + \frac{|0-1|}{2^4} + ... = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$
    3. $d[s,r] = \sum_{i=0}^{\infty} \frac{|s_i - r_i|}{2^i} = \frac{|0-1|}{1} + \frac{|0-0|}{2} + \frac{|1-1|}{2^2} + \frac{|0-0|}{2^3} + \frac{|0-1|}{2^4} + ... = 1 + \frac{0}{2} + \frac{0}{4} + \frac{0}{8} + \frac{1}{16} + ...$

2. Identify all sequences in $\sum_2$ which are periodic points of period 3 for σ. Which sequences lie on the same orbit under σ?

    – $s_1 = (100100100...), s_2 = (010010010...), s_3 = (001001001...)$
    – $s_4 = (110110110...), s_5 = (101101101...), s_6 = (011011011...)$

    když počítám i (00000...) a (11111...), tak mám pro periodu n 2^n bodů

3. Rework Exercise 2 for periods 4 and 5.
period 4:

    – $s_1 = (100010001000...), s_2 = (000100010001...), s_3 = (001000100010...), s_4 = (010001000100...)$

- $s_5 = (011101110111...), s_6 = (111011101110...), s_7 = (110111011101...)$, $s_8 = (101110111011...)$

period 5:

- $s_1 = (100001000010000...), s_2 = (000010000100001...), s_3 = (000100001000010...)$, $s_4 = (001000010000100...), s_5 = (010000100001000...)$
- $s_6 = (110001100011000...), s_7 = (100011000110001...), s_8 = (000110001100011...)$, $s_9 = (001100011000110...), s_{10} = (011000110001100...)$
- $s_{11} = (111001110011100...), s_{12} = (110011100111001...), s_{13} = (100111001110011...)$, $s_{14} = (001110011100111...), s_{15} = (011100111001110...)$
- $s_{16} = (111101111011110...)$ a posunutí (5x)
- $s_{21} = (001010010100101...)$ a posunutí (5x)

4. $\sum' \subset \sum_2 : s_j = 0 \Rightarrow s_{j+1} = 1$

   $s_0 = (101010...), s_1 = (110110110...), s_2 = (111011101110...)$,
   $s_3 = (111101111011110...)$
   $s_4 = (10110111011110...), s_5 = (1011010110...)$

   a. Show that σ preserves $\sum'$ and that $\sum'$ is a closed subset of $\sum$.
   zachování je jasné z definice - posunutím nemůžu dostat dvě nuly vedle sebe

   b. Show that periodic points of σ are dense in $\sum'$.
   vždycky najdu periodický bod libovolné periody

   c. Show that there is a dense orbit in $\sum'$.
   normálně v $\sum$ - dám za sebe postupně všechny možnosti bloků délky 1, 2, 3....
   (1 0 11 01 10 00 111 101 001 010 011 100 000 ...)

   v tomto prostoru stejně, akorát se omezím na prvky $\sum'$

   d. How may fixed points are there for $\sigma, \sigma^2, \sigma^3$ in $\sum'$?
   fixed points:
   σ - (11111111...)
   σ^2 - (101010...) a (010101...)
   σ^3 - $(110110110...), (101101101...)$ a (011011011...)

n-tá iterace má n pevných bodů asi, ale pravděpodobně to roste rychleji (1, 3, 4, 7, 11, 18...)

e. Find a recursive formula for the number of fixed points of σ^n in terms of the number of fixed points of σ^{n-1} and σ^{n-2}.
$\sigma^n = \sigma^{n-1} + \sigma^{n-2}$

5. $\sum_N$

a. N^n periodických bodů

b. stejným způsobem - naskládám za sebe všechny bloky

kapitola 1.7 - Topological conjugacy

1. Let $Q_c(x) = x^2 + c$. Prove, that if $c < \frac{1}{4}$, there is a unique $\mu > 1$ such that Q_c is topologically conjugate to $F_\mu(x) = \mu x(1-x)$ via a map of the form $h(x) = \alpha x + \beta$.

   topological conjugacy: $Q_c \circ h = h \circ F_\mu$

$$Q_c(h(x)) = h(F_\mu(x))$$

$$Q_c(\alpha x + \beta) = h(\mu x(1-x))$$
$$(\alpha x + \beta)^2 + c = \alpha \cdot (\mu x(1-x)) + \beta$$
$$\alpha^2 x^2 + 2\alpha\beta x + \beta^2 + c = \alpha\mu x - \alpha\mu x^2 + \beta$$
$$x^2(\alpha^2 + \alpha\mu) + x(2\alpha\beta - \alpha\mu) + (\beta^2 + c - \beta) = 0$$

$$\alpha^2 + \alpha\mu = 0 \quad \Rightarrow \alpha(\alpha + \mu) = 0 \quad \Rightarrow \alpha = 0 \vee \alpha = -\mu$$
$$2\alpha\beta - \alpha\mu = 0 \quad \Rightarrow \alpha(2\beta - \mu) = 0 \quad \Rightarrow \alpha = 0 \vee \beta = \frac{\mu}{2}$$
$$\beta^2 + c - \beta = 0 \quad \Rightarrow \beta = \frac{1 \pm \sqrt{1-4c}}{2}$$
$$c < \frac{1}{4} \Rightarrow \exists \beta \qquad \alpha, \beta \neq 0$$
$$\alpha = -\mu \qquad \beta = \frac{\mu}{2}$$

$$h(x) = -\mu x + \frac{\mu}{2}$$

2. A point p is a non-wandering point for f, if, for any open interval J containing p, there exists $x \in J$ and $n > 0$ such that $f^n(x) \in J$. Note that we do not require that p itself return to J. Let $\Omega(f)$ denote the set of non-wandering points for f.

a. Prove that $\Omega(f)$ is a closed set.
limitní body musí patřit do $\Omega(f)$.

b. If F_μ is the quadratic family with $\mu > 2+\sqrt{5}$, show that $\Omega(F_\mu) = \Lambda$ (Cantor set).
všechny ostatní body divergují k $-\infty$
$\mu > 2 + \sqrt{5} \Rightarrow \Lambda$ je konjugovaná se shiftem
every point is non-wandering
$x = 101001000\overline{1}$ je nebloudivý, ale není rekurentní (libovolný s periodickým rozvojem)
okolí $101 = \{x : x = 101...\} - y = \overline{101}$
okolí $101001 = \{x : x = 101001...\} - y = \overline{101001}$

c. Identify $\Omega(F_\mu)$ for each μ satisfying $0 < \mu \leq 3$.
two fixed points: x=0 (repelling), x=$\dfrac{\mu - 1}{\mu}$ (attracting)
pouze pevné body budou non-wandering

3. A point p is recurrent for f if, for any open interval J about p, there exists $n > 0$ such that $f^n(p) \in J$. Clearly, all periodic points are recurrent.

   a. Give an example of a non-periodic recurrent point for F_μ when $\mu > 2 + \sqrt{5}$.
   (asymptoticky periodický bod?) konjugace se shiftem:
   $x = 10|1|10|101|1011|...$ (začnu 10 a pak opisuju po blocích - nejvřív první číslo, potom první dvě,...)

   b. Give an example of a non-wandering point for F_μ which is not recurrent.
   (něco co konverguje k pevnému nebo periodickému bodu?) $x =$ 101001000100001... (najdu něco, co se k němu vrací, ale on se sám k sobě nevrátí)

kapitola 1.8 - Chaos

1. Prove that $F(x) = 4x^3 - 3x$ is chaotic on $[-1, 1]$.